ISRAEL JOURNAL OF MATHEMATICS 93 (1996), 185–188

A CONJECTURE ON THE NUMBER OF CONJUGACY CLASSES IN A p-SOLVABLE GROUP*

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ABSTRACT

If G is a p-solvable group, it is conjectured that $\mathrm{k}(G/\mathbb{O}_p(G)) \leq |G|_{p'}$. The conjecture is easily obtained for solvable groups as a consequence of R. Knörr's work on the $\mathrm{k}(GV)$ problem. Also, a related result is obtained: $\mathrm{k}(G/\mathbb{F}(G))$ is bounded by the index of a nilpotent injector of G.

1.

It is the aim of this note to raise the following question:

If p is a prime, G is a finite p-solvable group and $\mathrm{k}(G)$ is the number of conjugacy classes of G, is it true that $\mathrm{k}(G/\mathbb{O}_p(G)) \leq |G|_{p'}$?

The short proof of our Theorem A below shows that this is the case whenever G is solvable. This result is obtained as an easy consequence of Knörr's work on the $\mathrm{k}(GV)$-conjecture ([3]). The general problem seems to rely on the Simple Group Classification. (M. Isaacs, in private communication, has given a positive answer to the question above for groups with abelian Sylow p-subgroups without using the classification, however.)

Also, a related result is shown in our Theorem B: if G is a solvable group, then $\mathrm{k}(G/\mathbf{F}(\mathrm{G}))$ is bounded by the index of a nilpotent injector of G.

These two results combined suggest that, perhaps, there is an important number of Fitting classes $\mathcal{F}$ for which, in solvable groups, $\mathrm{k}(G/G_{\mathcal{F}})$ is bounded by the index of an $\mathcal{F}$-injector.

* Research partially supported by DGICYT.PB 90-0414-C02-01.
Received October 18, 1994

2.

THEOREM A: *If G is a solvable group, then* $\mathrm{k}(G/\mathbb{O}_p(G)) \leq |G|_{p'}$.

We need a well known result on conjugacy classes due to P. X. Gallagher.

LEMMA: *If G is a finite group and $N \triangleleft G$, then* $\mathrm{k}(G) \leq \mathrm{k}(G/N)\mathrm{k}(N)$.

Proof: See [1]. ∎

Proof of Theorem A: We argue by induction on $|G|$.

It is clear that we may assume $\mathbb{O}_p(G) = 1$, so we are forced to prove that $\mathrm{k}(G) \leq |G|_{p'}$.

If $1 < N$ is a normal subgroup of G and $M/N = \mathbb{O}_p(G/N)$, by the inductive hypothesis, we have that

$$\mathrm{k}(G/M) \leq |G/N|_{p'}.$$

Now, if $M < G$, again by induction, we have that

$$\mathrm{k}(M) \leq |N|_{p'}$$

since $\mathbb{O}_p(M) = 1$ and $|M|_{p'} = |N|_{p'}$. Therefore, if M is proper in G, the theorem follows from the lemma.

So we may assume that $G = VP$, where V is a normal elementary abelian q-group of G (q a prime different from p) and P is a Sylow p-subgroup. Now, V is a faithful $\mathrm{GF}(q)[P]$-module and by R. Knörr's theorem (7.4 of [3]), it follows that

$$\mathrm{k}(G) = \mathrm{k}(VP) \leq |V| = |G|_{p'}$$

as required. ∎

3.

The aim of this section is to prove Theorem B.

THEOREM B: *Let G be a solvable group and let I be a nilpotent injector of G. Then* $\mathrm{k}(G/\mathbf{F}(G)) \leq |G\colon I|$.

This will be obtained as a consequence of the following result.

THEOREM: *Let G be solvable and write $F = \mathbb{F}(G)$ and $Z = \mathbb{Z}(F)$. Let I be a nilpotent injector of G. Then $\mathrm{k}(G/Z) \le |G: I|\mathrm{k}(F/Z)$.*

Proof: We argue by induction on $|G|$. We freely use the results of A. Mann in [4]. Write $C_p = \mathbb{C}_G(F_{p'})$ for each prime divisor p of F. We have that $C_p \triangleleft G$ and that C_p contains the Sylow p-subgroup of I. Thus $N = \prod C_p$ is a normal subgroup of G containing I. In particular, $F = \mathbb{F}(N)$ and I is a nilpotent injector of N. If $N < G$, induction yields that $\mathrm{k}(N/Z) \le |N: I|\mathrm{k}(F/Z)$. Since $\mathrm{k}(G/Z) \le \mathrm{k}(N/Z)|G: N|$, the result follows in this case. We can thus assume that $\prod C_p = G$.

Observe that $Z \subseteq C_p$ for all p. We claim that in fact $C_p \cap \prod_{q\neq p} C_q = Z$. To see this, it suffices to observe that C_p centralizes $F_{p'}$ and the product centralizes F_p, and so the intersection centralizes F. Since $C_G(F) \subseteq F$, the claim is proved. Hence, it follows that the group G/Z is the internal direct product of its subgroups C_p/Z and therefore $\mathrm{k}(G/Z) = \prod \mathrm{k}(C_p/Z)$.

If $C_p = G$ for some prime p, then $F_{p'} \subseteq \mathbb{Z}(G)$. But $I_{p'}$ centralizes F_p and hence $I_{p'} \subseteq \mathbb{C}_G(F) \subseteq F$. Thus I/F is a p-group and $|G/F|_{p'} \le |G: I|$. In this case, $F/Z = \mathbb{O}_p(G/Z)$, and so by Theorem A, we have $\mathrm{k}(G/F) \le |G/F|_{p'} \le |G: I|$. Thus

$$\mathrm{k}(G/Z) \le \mathrm{k}(G/F)\mathrm{k}(F/Z) \le |G: I|\mathrm{k}(F/Z),$$

as required. We can thus assume that $C_p < G$ for all primes p.

Write $E_p = \mathbb{F}(C_p) = C_p \cap F$ and observe that $F_p \subseteq E_p$. It follows that $Z = \mathbb{Z}(E_p)$. Now $C_p \cap I = J_p$ is a nilpotent injector of C_p. The inductive hypothesis applied to C_p yields that

$$\mathrm{k}(G/Z) = \prod \mathrm{k}(C_p/Z) \le \prod |C_p: J_p| \prod \mathrm{k}(E_p/Z)$$

and our task now is to estimate each of the products on the right, above.

Recall that G/Z is the direct product of the subgroups C_p/Z. Write $E = \prod E_p$ and note that E/Z is the direct product of its subgroups E_p/Z. Thus $\prod \mathrm{k}(E_p/Z) = \mathrm{k}(E/Z)$. But $E \subseteq F$ and in fact $E = F$ since E_p contains F_p. Thus $\prod \mathrm{k}(E_p/Z) = \mathrm{k}(F/Z)$.

Similarly, write $J = \prod J_p$ and note that J is a subgroup and that J/Z is the direct product of its subgroups J_p/Z. It follows that $\prod |C_p: J_p| = |G: J|$. Since the Sylow p-subgroup of I is contained in C_p, it is contained in J_p, and it follows that $I \subseteq J$. Thus $\prod |C_p: J_p| \le |G: I|$. The result now follows. ∎

Proof of Theorem B: Apply the theorem above to the group $\overline{G} = G/\Phi(G)$. Note that $\overline{F} = \mathbb{F}(\overline{G})$ is abelian and that $\overline{I}$ is contained in a nilpotent injector $\overline{J}$ of $\overline{G}$, where $I \subseteq J \subseteq G$. The theorem then yields

$$\mathrm{k}(G/F) = \mathrm{k}(\overline{G}/\overline{F}) \leq |\overline{G}\colon \overline{J}| = |G\colon J| \leq |G\colon I|,$$

as required. ∎

There are two corollaries on character theory that, perhaps, are worth mentioning.

COROLLARY A: *Suppose that G is a solvable group and let $\theta \in \mathrm{Irr}(\mathbb{O}_p(G))$ be G-invariant. Then $|\mathrm{Irr}(G|\theta)| \leq |G|_{p'}$.*

COROLLARY B: *Suppose that G is a solvable group and let $\theta \in \mathrm{Irr}(\mathbb{F}(G))$ be G-invariant. Then $|\mathrm{Irr}(G|\theta)| \leq |G\colon I|$, where I is a nilpotent injector of G.*

Proof of Corollaries A and B: Recall that if $\theta \in \mathrm{Irr}(N)$ is G-invariant, then $|\mathrm{Irr}(G|\theta)|$ is the number of the so-called θ-special conjugacy classes of G/N, by Problem (6.10) of [2]. Now, the result follows by applying Theorems A and B. ∎

References

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